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Nuclear Field Theory with Chiral Symmetry on a Calabi–Yau Manifold

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The purpose of this contribution is to show how a nuclear field theory follows naturally from the structure of four-dimensional Riemannian geometry. A Yang–Mills field is introduced by constructing fibers that include all possible exchanges of spin, parity, and charge such that the collective quantum numbers remain the same. In this way $O(4)$ internal symmetry transformations are found and a connection is obtained by exponentiation of a CP-invariant operator C associated with the ground state. The metric is Calabi–Yau and Einstein. Carbon-13 is chosen as an example because it is the lightest nucleus to exhibit small spin mutations even though there is no deformation parameter in the $O(4)$ commutation relations. Instead, a supersymmetric transformation replaces a quantum group. Mirror symmetry is also discussed and because a density functional approach is used it is possible to regard the nucleus as a statistical ensemble.

1. INTRODUCTION

de Wet (1996) considered an example of how a Z_2-graded algebra, specifically the Lie algebra of $O(4)$, leads naturally to the well-known angular momentum matrices σ_i of a coupled system of P protons and N neutrons, namely

$$\sigma_i = E_N \otimes {}^P\Gamma_i + {}^N\Gamma_i \otimes E_P, \qquad i = 1,2,3 \tag{1.1}$$

where ${}^P\Gamma_i$, ${}^N\Gamma_i$ are $(P + 1)$-, $(N + 1)$-dimensional Lie operators of $so(3)$; E_P, E_N are $(P + 1)$, $(N + 1)$ unit matrices.

Essentially a Z_2-graded algebra splits a bundle Λ^2 into the direct sum

$$\Lambda^2 = \Lambda^2_+ + \Lambda^2_- \tag{1.2}$$

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of self-dual and anti-self-dual 2-forms, respectively. An example of this grading is the decomposition

$$so(4) \cong so(3) + so(3) \tag{1.3}$$

into bundles of three-dimensional Lie algebras which were long ago identified by de Wet (1971) with spin and isospin (based on some ideas of Eddington). In a seminal paper Atiyah *et al.* (1978) use this decomposition on the Lie group level to introduce, at least locally, the two complex spinor bundles V_+ and V_-; the bundles of self-dual and anti-self-dual spinors. Then $V = V_+ + V_-$ is isomorphic to the complexified Clifford algebra bundle of one forms Λ^1 [Eddington (1948) called the Clifford algebra C_4 a Sedenion algebra and we will use his transparent Sedenion, or E-number, notation].

The purpose of this contribution is to show how a nuclear field theory follows naturally from the structure of four-dimensional Riemannian geometry and to this end we shall consider the Hodge star mapping

$$*: \quad \Lambda^2 \to \Lambda^2 \tag{1.4}$$

as transforming a nucleus into its mirror image, i.e., $(P, N) \to (N, P)$.

Under these conditions

$$\text{spin}(\sigma) \to \text{spin}(\sigma); \qquad \text{isospin}(T_3) \to \text{-isospin}(T_3) \tag{1.5}$$

are the self-dual and anti-self-dual forms. An example is given by the first and fourth columns of Table I of Section 3. [Here we have denoted parity by p and the spin by s, and in Section 2 we shall see how the nuclear charge-spin-parity states are labeled by the partition $[\lambda_1\lambda_2\lambda_3\lambda_4]$ of $(A = N + P)$ and its four permutations that appear in (2.6a).]

Furthermore, Atiyah *et al.* (1978) consider the decomposition of the complexified Clifford algebra bundle

$$\Lambda^1 = \Lambda_c^0 + \Lambda_c^1$$

such that the image of V_- in Λ_c^1 is the subspace $\Lambda^{1,0}$ of (1,0) forms that define a complex structure of the kind considered by de Wet (1995, 1996). Now a complex manifold in turn decomposes into a sum of the spaces $\Lambda^{1,0}$ and $\Lambda^{0,1}$ of (1,0) and (0,1) forms (Kobayashi and Nomizu, 1969, Chapter IX), so it is natural to identify the space $\Lambda^{0,1}$ with the self-dual form σ associated with V_+. Then its conjugate

$$\pi_i = E_N \otimes {}^P\Gamma_i - {}^N\Gamma_i \otimes E_P \qquad (i = 1,2,3) \tag{1.6}$$

lies in $\Lambda^{1,0}$ (Kobayashi and Nomizu, 1969). We shall see in Section 2 how π is parity, but for the moment simply observe that this definition is also consistent with Table I as described in Section 3. The six operators σ_i, π_i are generators of $O(4)$.

Now a complex structure occurs only on fermions (odd A), the even A nuclei being characterized by Euclidean structure, and an example of the decomposition of the two complex manifolds carrying ^{9}Li, ^{9}C is

$$^9\text{Li:}\quad 6C_{[3303]} = 34(\sigma + \pi) + 9(\sigma\pi^2 + \sigma^2\pi) + (\sigma^3 + \pi^3) \quad (1.7a)$$

$$^9\text{C:}\quad 6C_{[3033]} = 34(\sigma - \pi) + 9\,(\sigma\pi^{\,2} - \sigma^2\pi) + (\sigma^3 - \pi^3) \quad (1.7b)$$

which is manifestly CP-symmetric because $T_3 \rightarrow -T_3$ is accompanied by $\pi \rightarrow -\pi$. Equation (1.7a) confirms the decompositions given by Kobayashi and Nomizu (1969) and Salamon (1989), where the Wigner coefficients are the number of times the irreducible spin representations

$$S^{1,1} = (\sigma + \pi), \qquad S^{2,1} = (\sigma\pi^2 + \sigma^2\pi), \qquad S^{3,0} = (\sigma^3 + \pi^3) \qquad (1.8)$$

are contained in the subspaces $\Lambda^{1,1}$, $\Lambda^{2,1}$, $\Lambda^{3,0}$ of Λ^3 which are embedded in the Clifford algebra of the A coordinates of σ,π and their products [cf. Lawson and Michelsohn (1989) for the isomorphism between Clifford algebras and exterior products].

An inspection of (1.7a) and (1.7b) shows clearly that fermion CP invariance follows from the decompositions $S^{1,1}$, $S^{2,1}$, $S^{3,0}$ of the complex manifold. Moreover, since such a decomposition applies only to the state [3303] we will associate the ground state with the label $[\Lambda] \equiv [\Lambda_1\Lambda_2\Lambda_3\Lambda_4]$. Then higher energy states will be labeled by $[\lambda] = [\lambda_1\lambda_2\lambda_3\lambda_4]$. These, however, are characterized by nuclear decay, so are no longer in the same manifold.

In Section 3 it will be shown that the fermion manifolds have a Ricci-flat Kaehler metric and are therefore Calabi–Yau. Recently there have been several studies of the mirror symmetry of Calabi–Yau spaces (e.g., Strominger *et al.*, 1996), but the mirror nuclear manifolds appear to be isomorphic. For example, in Section 3 the matrix representations of the CP-invariant operators $C_{[4324]}$, $C_{[4234]}$ of, respectively, ^{13}C, ^{13}N are identical up to interchange of rows and columns. These representations are derived by substituting (1.1), (1.6) into the equations (3.8a) and (3.8b) [which are analogues of (1.7a) and (1.7b)] and their rotational eigenvalues $C'_{[\lambda]}$ appear in the last column of Table I. However, we can substitute directly in (3.8a) and (3.8b) using (3.9), which is derived from a canonical labeling scheme suggested by (2.6a) of Section 2. Again this labeling gives rise to an isomorphism with almost identical rotational eigenvalues $C_{[\lambda]}$ in the penultimate column of Table I.

In fact there are only tiny spin mutations (marked by asterisks) associated with the states [2533], [4333] of ^{13}C. As discussed in Section 3, these are believed to be due to Yang–Mills interaction even though the group $O(4)$ has no deformation parameter q in its commutation relations and is not a quantum group. Instead, interaction simply changes the spins of two neutrons in paired states, so we have replaced quantum group theory by supersymmetry!

In line with the aims of this contribution we have outlined several correspondences between nuclear theory and the structure of Z_2-graded algebras, which of course also play a role in quantum group theory as outlined by Manin (1991, Chapter 4). However, because the left ideals (2.3) of the Dirac ring are bra vectors generated by a density operator $\Psi^{(A)}$ (Biedenharn and Louck, 1981, § 7.7), we may also consider our model as a canonical ensemble of nucleons (Prigogine, 1995). In fact it will appear that the partition (2.5) is precisely into the states:

λ_1 = number of neutrons with a positive spin and negative parity
λ_2 = number of neutrons with a negative spin and positive parity
λ_3 = number of protons with a negative spin and negative parity
λ_4 = number of protons with a positive spin and positive parity

of a canonical ensemble.

Then the self-dual and anti-self-dual forms are the pairs $\{\lambda_1; \lambda_4\}$, $\{\lambda_2; \lambda_3\}$, and each of the points on the microcanonical surface has the same probability of representing the system. In particular the radii of surface curvature should be uniformly distributed and this can be verified by actually calculating sectional curvatures from the metrics of Section 3.

We can now move on to an outline of how a Yang–Mills field is incorporated.

2. FIELD THEORY

The basic theory has been reviewed in Section 1 of de Wet (1994), so only an outline will be given here. The method used constructs tensor products in the enveloping algebra $A(\gamma)$ of the Dirac ring of an irreducible self-representation

$$\frac{1}{4}\Psi = (iE_4\psi_1 + E_{23}\psi_2 + E_{14}\psi_3 + E_{05}\psi_4)e \tag{2.1}$$

with itself. Here Eddington's E-numbers are related to the Dirac matrices by

$$\gamma_\nu = iE_{\sigma\nu}, \qquad E_{\mu\nu} = E_{\sigma\mu}E_{\sigma\nu}, \qquad E^2_{\mu\nu} = -1,$$
$$E_{\mu\nu} = -E_{\nu\mu}, \qquad \mu < \nu = 1, \ldots, 5$$

and the commuting operators E_{23}, E_{14}, and E_{05} are, respectively, independent infinitesimal rotations in 3-space, 4-space, and isospace that correspond to the spin σ, parity π, and charge T_3 carried by a single nucleon. The parameters ψ_2, ψ_3, ψ_4 are half-angles of rotation and e is a primitive idempotent of the Dirac ring; E_4 is the unit matrix.

A rotation through 180° about x will change spin up to spin down and if this is followed by a rotation of 180° about t, x can go to $-x$ without inverting time, but instead changing to a left-handed coordinate system. Thus we associate the rotation E_{14} about x in 4-space with a parity reversal $E_{14} \to -E_{14}$, and this way the time coordinate is 'rolled up' so that the Lorentz-invariant representation (2.1) can describe a nucleon in 3-space.

The basis elements of $A(\gamma)$ are the $4^A \times 4^A$ matrices ($A = N + Z$)

$$E^l_{\mu\nu} = E_4 \otimes \ldots \otimes E_4 \otimes E_{\mu\nu} \otimes E_4 \otimes \ldots \otimes E_4$$

with $E_{\mu\nu}$ in the lth position. The elements $E^l_{\mu\nu}$, $E^{l+1}_{\mu\nu}$ commute, and $A(\gamma)$ is found to have the following generators:

$$\Gamma^{(A)}_\nu = \frac{1}{2}(E^1_{0\nu} + E^2_{0\nu} + \ldots + E^A_{0\nu}), \qquad \nu = 1, \ldots, 5 \tag{2.2a}$$

$$\sigma^{(A)}_{\mu\nu} = [\Gamma^{(A)}_\mu, \Gamma^{(A)}_\nu] = \frac{1}{2}(E^1_{\mu\nu} + E^2_{\mu\nu} + \ldots + E^A_{\mu\nu}) \tag{2.2b}$$

$$\eta^{(A)}_\nu = E_{0\nu} \otimes \ldots \otimes E_{0\nu} = E^1_{0\nu} E^2_{0\nu} \ldots E^A_{0\nu} \tag{2.2c}$$

$$\eta^{(A)}_{\mu\nu} = \eta^{(A)}_\mu \eta^{(A)}_\nu = E^1_{\mu\nu} E^2_{\mu\nu} \ldots E^A_{\mu\nu}, \qquad \mu < \nu = 1,\ldots,5 \tag{2.2d}$$

Then the irreducible representations, or minimal left ideals, of $A(\gamma)$ are

$$\Psi^{(A)} = \sum_\lambda C_{[\lambda]} P_{[\lambda]} \tag{2.3}$$

with

$$C_{[\lambda]} = i^{\lambda_1} C(E^1_{23} \ldots E^{\lambda_2}_{23} E^{\lambda_2+1}_{14} \ldots E^{\lambda_2+\lambda_3}_{14} E^{\lambda_2+\lambda_3+1}_{05} \ldots E^{A-\lambda_1}_{05}) \tag{2.4}$$

if C denotes summation over the $N_{[\lambda]} = A!/(\lambda_1!\lambda_2!\lambda_3!\lambda_4!)$ combinations of the basis elements appearing in the bracket. Here $[\lambda] \equiv [\lambda_1\lambda_2\lambda_3\lambda_4]$ is a partition

$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = A \tag{2.5}$$

and

$$\begin{aligned} P_{[\lambda]} = i^{-A} (& i^A \psi_1^{\lambda_1} \psi_2^{\lambda_2} \psi_3^{\lambda_3} \psi_4^{\lambda_4} \\ & + \eta^{(A)}_{23} \psi_2^{\lambda_1} \psi_1^{\lambda_2} \psi_4^{\lambda_3} \psi_3^{\lambda_4} + \eta^{(A)}_{14} \psi_3^{\lambda_1} \psi_4^{\lambda_2} \psi_1^{\lambda_3} \psi_2^{\lambda_4} \\ & + \eta^{(A)}_5 \psi_4^{\lambda_1} \psi_3^{\lambda_2} \psi_2^{\lambda_3} \psi_1^{\lambda_4}) \, \varepsilon_A \end{aligned} \tag{2.6a}$$

is a projection operator satisfying

$$P^2_{[\lambda]} = P_{[\lambda]} \psi, \qquad \psi \equiv \psi_1\psi_2\psi_3\psi_4 \tag{2.6b}$$

Also, $\varepsilon_\lambda = e \otimes \ldots \otimes e = e^1 e^2 \ldots e^A$ is a primitive idempotent in $A(\gamma)$, so that (2.6a) has the same form for A nucleons as the basic representation (2.1).

By studying (2.6a), it can be shown that a canonical labeling scheme associates $(\lambda_3 + \lambda_4)$ with the number of nucleons with a positive charge (i.e., the first two terms represent a nucleus and the last two terms its mirror image), $(\lambda_2 + \lambda_3)$ the number with a given spin σ, and $(\lambda_2 + \lambda_4)$ the number with a particular parity π. Thus each partition (2.5) represents a charge-spin-parity state of a nucleus, and by choosing 4×4 matrix representations for E_{23}, E_{14}, E_{03} and constructing fibers that include every possible exchange of spin, parity, and charge between nucleons, such that the collective quantum numbers remain the same, it may be shown that $C_{[\lambda]}$ partitions beautifully into a de Rham decomposition of isobaric multiplets. Under these conditions the $4^A \times 4^A$ matrices (2.2) shrink to (1.1), (1.6) with rows labeled by the fibers $[\lambda]$, where $\sigma_1 \equiv \sigma_{23}^{(A)}$, $\pi_1 \equiv \sigma_{14}^{(A)}$ are two of the six generators of $O(4)$.

In this way the nucleons interact by means of a Yang–Mills gauge field which can be determined by calculating the connections in the fiber bundle. This has been done by de Wet (1966) by exponentiating $C_{[\Lambda]}$ and finding the Ricci-flat Kaehler metric of the resulting Calabi–Yau space or torus. In Section 3 this will be shown to be Einstein, which ties in with the ideas of Capovilla *et al.* (1990) that, say, that an $SU(2)$ connection characterizes a solution of the source-free Einstein field equations. In fact, a compact 4-manifold acted on by the group $SU(2)$ must be a Ricci-flat torus (Salamon, 1989, p. 106). In other words, the nuclear metric is a solution of the source-free Einstein equations!

From another point of view we can regard the Dirac algebra as the infinitesimal ring of Minkowski space and therefore as a tangent space to 4-dimensional space-time in the spirit of Ashtekar (1988). The representations of the tangent space that give us the internal symmetries σ, π, T are by construction a soldering form (Ashtekar *et al.*, (1988)) and exponentiation must necessarily take us back to source-free Einstein space.

Returning to (2.4), the bases of the form $\Lambda^{\lambda_2,\lambda_3}$ are contained in $C_{[\lambda]}$ without the E_{05} elements, which, as we shall see, are needed only to characterize a particular member of an isobaric multiplet. Thus in the next section, where an outline of the decomposition (1.7) is given, it will become clear that a new (p,q) subspace appears whenever the products $\sigma_0\pi_0$ of

$$\sigma_0 = 2\sigma_1 = (E_{23}^1 + \ldots + E_{23}^A), \qquad \pi_0 = 2\pi_1 = (E_{14}^1 + \ldots + {}_{14}^A) \tag{2.7}$$

contain terms with the same indices. Under these conditions $p + q \leq \lambda_2 + \lambda_3$.

Although a general nuclear state is labeled by $[\lambda]$, there is only one state $[\Lambda] = [\Lambda_1\Lambda_2\Lambda_3\Lambda_4]$ having the decomposition (1.7) associated with the ground state. Then $[\Lambda]$ itself carries all the spin-parity states $[\lambda]$ of Table I. These label the rows of an irreducible submatrix

$$\mu = \begin{bmatrix} & B \\ -B^t & \end{bmatrix} \tag{2.8}$$

of $C_{[\Lambda]}$ which has the holomorphic coordinates $z_k = \pm i\lambda_k$, where λ_k is the eigenvalue associated with $[\lambda]_k$ by means of the correspondence (3.9) and is real for the submatrix B.

In fact z_k, $\bar{z}_k$, characterize the horizontal subspace of a complex Grassmann or Kaehler manifold (Kobayashi and Nomizu, 1969, Chapter IX) and because it is also Ricci-flat and Kaehler it is a twistor space (using the definition of Lawson and Michelsohn, 1989, Chapter IV, Section 9). We shall see in Section 3 how a metric is obtained.

3. AN EXAMPLE; CARBON-9, 13

In this section the ideas outlined above will be brought together with an example that exhibits spin mutation and at the same time illustrates in more detail how the decomposition (1.7) of a complex manifold is obtained. We begin by replacing (2.4) with

$$C_{[\Lambda]} = i^{\Lambda_1}\sigma_0^{\Lambda_2}\pi_0^{\Lambda_3}T_0^{\Lambda_4}. - \sum_{\lambda} i^{\lambda_1}\sigma_0^{\lambda_2}\pi_0^{\lambda_3}T_0^{\lambda_4} \tag{3.1}$$

where in addition to (2.7)

$$T_0 = 2\Gamma_5^{(A)} = (E_{05}^1 + \ldots + E_{05}^A)$$

The real quantum numbers s, p, and $T_3 = 1/2\ (Z - N)$ of spin, parity, and charge are

$$\sigma_0 = 2is, \qquad \pi_0 = 2iP, \qquad T_0 = 2iT_3 = i(Z - N) \tag{3.2}$$

which show how the quantum numbers of individual nucleons are additive.

The summation contains all those terms arising from repeated indices $E_{23}^jE_{23}^j$; $E_{23}^jE_{14}^j$; $E_{23}^jE_{05}^j$; $E_{14}^jE_{05}^j$ that yield a single term according to the multiplication table

	E_{23}^j	E_{14}^j	E_{05}^j
E_{23}^j	i^2	iE_{05}^j	iE_{14}^j
E_{14}^j	iE_{05}^j	i^2	iE_{23}^j
E_{05}^j	iE_{14}^j	iE_{23}^j	i^2

(3.3)

An elementary example is

$$\sigma_0T_0 = P(E_{23}^i\ E_{05}^j) + i\pi_0 \tag{3.4a}$$

where P denotes summation over the $A!/(A-n)!$ permutations of the n generators in the bracket. Then

$$C_{[(A-2)101]} = i^{(A-2)}P(E_{23}^iE_{05}^j) = i^{(A-2)}(\sigma_0T_0 - i\pi_0) \tag{3.4b}$$

and if $A = 3$, $Z = 1$, $T = -i$, so

$$C_{[1101]} = (\sigma_0 + \pi_0) \tag{3.4c}$$

which characterizes the ground state of ^3H. Now interchange $\sigma_0 \leftrightarrow \pi_0$ in (3.4b) to get

$$C_{[(A-2)011]} = i^{(A-2)}(\pi_0 T_0 - i\sigma_0) \tag{3.4d}$$

Then if $A = 3$, $Z = 2$, $T = i$, we have

$$C_{[1011]} = (\sigma_0 - \pi_0) \tag{3.4e}$$

which charaterizes the ground state of ^{3}He. Equation (3.4c) is the irreducible spin representation Λ^1 of (1.8), which occurs once only, and (3.4a) is an example of a single term π_0 arising from $E^i_{14} = E^i_{23}\, E^i_{05}$ $(i = 1, \ldots, A)$. Because T_0 is a scalar, this term dictates the size of the subspace Λ^1.

Let us now 'add' another nucleon by multiplying (3.4a) by $\pi_0 = (E^1_{14} + \ldots + E^A_{14})$ to obtain

$$\sigma_0\, \pi_0\, T_0 = P(E^i_{23}\, E^j_{14}\, E^k_{05}) + i\{P(E^i_{23}\, E^j_{23}) + P(E^i_{14} E^j_{14}) + P(E^i_{05}\, E^j_{05})\} + Ai^3 \tag{3.5a}$$

where

$$\begin{aligned} \sigma_0^2 &= P(E^i_{23}\, E^j_{23}) + Ai^2; \\ \pi_0^2 &= P(E^i_{14}\, E^j_{14}) + Ai^2, \\ T_0^2 &= P(E^i_{05}\, E^j_{05}) + Ai^2 \end{aligned} \tag{3.5b}$$

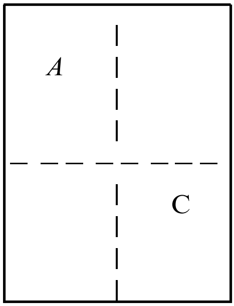
Thus

$$\begin{aligned} C_{[(A-3)111]} &= i^{(A-3)} P(E^i_{23} E^j_{14} E^k_{05}) \\ &= i^{(A-3)}[\sigma_0\, \pi_0\, T - i(\sigma_0^2 + \pi_0^2 + T_0^2 - 3Ai^2) - Ai^3] \end{aligned} \tag{3.5c}$$

Then if $A = 4, Z = 2$, $T_0 = 0$,

$$C_{[1111]} = \sigma_0^2 + \pi_0^2 + 8 \tag{3.5d}$$

which characterizes the ground state of ^{4}He found to have only one spin configuration. In this case there is no mirror nucleus and A is even, so there is no decomposition like that of (1.7). We are in fact in a vertical subspace h of the tangent space to the boson manifold with a matrix representation



(3.6)

Clearly the process may be continued until ultimately the invariant operator for ^{9}Li is

$$C_{[3303]} = i^3\, P(E^i_{23}\, E^j_{23}\, E^k_{23}\, E^l_{05}\, E^m_{05}\, E^n_{05})/(3!\ 3!) \tag{3.7}$$

which yields (1.7a) after writing $T_0 = i(Z - N) = -3i$ and making use of subsidiary relations such as (3.5b). When $A = 13$ we find

$$\begin{aligned} {}^{13}\text{C:}\quad -12C_{[4324]} &= 3089\,(\sigma_0 - \pi_0) + 151(\sigma_0\, \pi_0^2 - \sigma_0^2\pi_0) \\ &\quad + 135\,(\sigma_0^3 - \pi_0^3) + 3(\sigma_0^3\pi_0^2 - \sigma_0^2\pi_0^3) \\ &\quad + (\sigma_0\, \pi_0^4 - \sigma_0^4\pi_0) + (\sigma_0^5 - \pi_0^5) \end{aligned} \tag{3.8a}$$

$$\begin{aligned} {}^{13}\text{N:}\quad -12C_{[4234]} &= 3089(\sigma_0 + \pi_0) + 151(\sigma_0\, \pi_0^2 + \sigma_0^2\, \pi_0) \\ &\quad + 135(\sigma_0^3 + \pi_0^3) + 3(\sigma_0^3\pi_0^2 + \sigma_0^2\pi_0^3) \\ &\quad + (\sigma_0\, \pi_0^4 + \sigma_0^4\pi_0) + (\sigma_0^5 + \pi_0^5) \end{aligned} \tag{3.8b}$$

and once again we have precisely the decomposition given by Salamon (1989, p. 33) of $\Lambda^5 = \Lambda^{\Lambda_2+\Lambda_3}$.

Now if we assume, in accord with the canonical labeling suggested by (2.6a), that $(\lambda_2 + \lambda_3)$ is the number of nucleons with negative spin σ and $(\lambda_2 + \lambda_4)$ the number with positive parity π, then we can determine the eigenvalues of $C_{[\Lambda]}$ associated with each configuration $[\lambda]$ simply by substitution of

$$\sigma_0 = \{A - 2(\lambda_2 + \lambda_3)\}\, i, \qquad \pi_0 = \{2(\lambda_2 + \lambda_4) - A\}i \tag{3.9}$$

in (3.8). These are eigenvalues without any interaction because as yet no use has been made of (1.1), (1.6). However, we can also substitute directly from these equations [remembering from (2.7) that $\sigma_0 = 2\sigma_1$, $\pi_0 = 2\pi_1$] and use the standard representations of $so(3)$ for Γ_1 to find a matrix representation μ of $C'_{[\Lambda]}$. The matrix representations of ^{13}N and ^{13}C are identical up to an exchange of rows and columns and Table I (which does not show repeated eigenvalues) compares the eigenvalues of $C_{[\Lambda]}$ and $C'_{[\Lambda]}$. Because of the parity change columns 1 and 4 will also yield the same eigenvalues, as will columns 2 and 3 up to a sign change caused by $\sigma_0 \rightarrow -\sigma_0$. Those states associated with the matrix representation $C'_{[\Lambda]}$ in the last column are marked by an asterisk and have repeated eigenvalues except when $\lambda_3 = \lambda_4 = 3$.

It is apparent that only the spin-parity states [2333], [4333] exhibit a tiny mutation of 2/900. However, if the eigenvalues of these states are interchanged so that the ground state [4333] has the eigenvalue −460 instead of −468, and [2533] assumes 468, not 460, the mutations disappear. Thus these two states are paired, differing only in the number of neutrons with negative spin, so the introduction of a Yang–Mills field simply changes the spin of two neutrons in the paired states, which amounts to a supersymmetric transformation. Another example of paired states is given by Fig. 1 of de

Table I. Coherent States of ^{13}C, ^{13}N

^{13}C s + p −	^{13}C s − p +	^{13}N s − p −	^{13}N s + p +	^{13}C		^{13}N			Matrix representation
$\lambda_1\lambda_2\lambda_3\lambda_4$	$\lambda_2\lambda_1\lambda_4\lambda_3$	$\lambda_3\lambda_4\lambda_1\lambda_2$	$\lambda_4\lambda_3\lambda_2\lambda_1$	σ_0	π_0	σ_0	π_0	$(C_{[\lambda]} + 3500)$ 3600	$C'_{[\lambda]} + 3500$ 3600
7006	0760*	0670*	6007	$13i$	$-i$	$13i$	i	35/9	35/9
7015*	0751	1570	5107*	$11i$	$-3i$	$11i$	$-3i$	0	0
7024	0742*	2470*	4207	$9i$	$-5i$	$9i$	$5i$	1.4	1.4
7033*	0733	3370	3307*	$7i$	$-7i$	$7i$	$7i$	56/90	56/90
6106*	1660	0661	6016*	$11i$	i	$11i$	$-i$	5/9	5/9
6115	1651*	1561*	5116	$9i$	$-i$	$9i$	i	2/3	2/3
6124*	1642	2461	4216*	$7i$	$-3i$	$7i$	$3i$	114/90	114/90
6133	1633*	3361*	3316	$5i$	$-5i$	$5i$	$5i$	68/90	68/90
5206	2560*	0652*	6025	$9i$	$3i$	$9i$	$-3i$	1	1
5215*	2551	1552	5125*	$7i$	i	$7i$	$-i$	10/9	10/9
5224	2542*	2452*	4225	$5i$	$-i$	$5i$	i	1	1
5233	2533*	3352*	3325	$3i$	$-3i$	$3i$	$3i$	99/90*	99.2/90
4306*	3460	0643	6034*	$7i$	$5i$	$7i$	$-5i$	86 /90	86 /90
4315	3451*	1543*	5134	$5i$	$3i$	$5i$	$-3i$	88/90	88 /90
Λ4324*	3442	2443	4234*	$3i$	i	$3i$	$-i$	79.6/90	79.6/90
gs4333*	3433	3343	3334*	i	$-i$	i	i	75.8/90*	76 /90

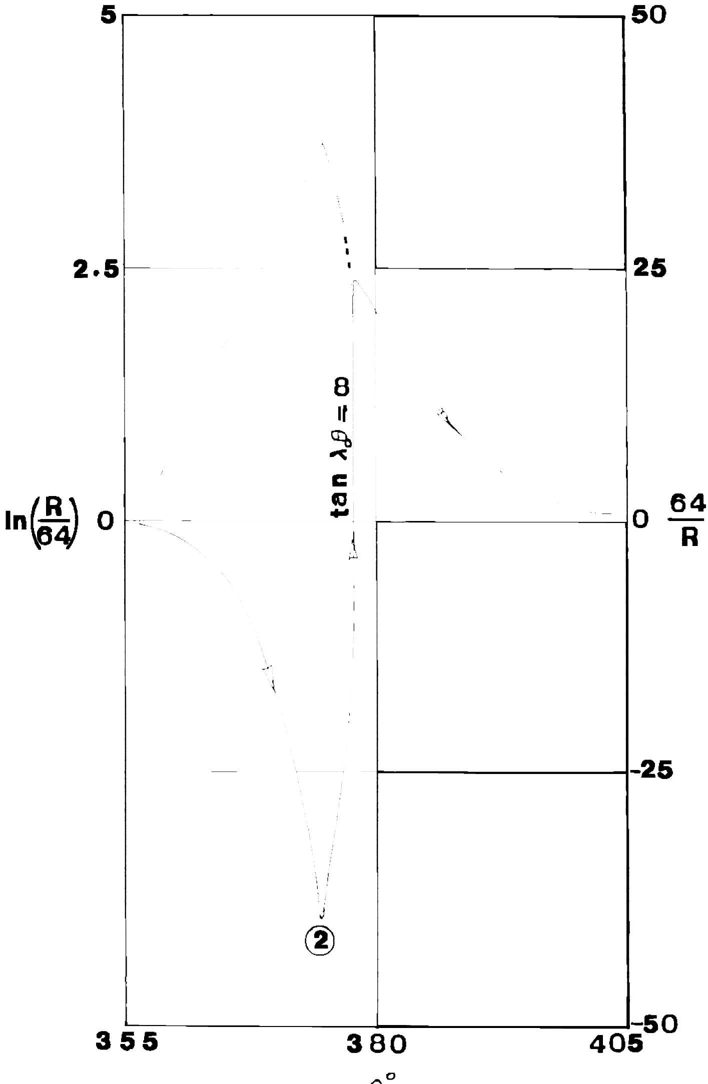


Fig. 1. Instanton on ^{9}C manifold.

Wet (1995, 1996), where X_4 is the ground state [3033] of ^{9}Li and X_5 could be the state [3321].

Yang–Mills fields do not change the energy, so there can be no dissipation due to spin mutations (this would lead to the collapse of nuclei to a zero-spin state). Thus there must either be supersymmetry or the mutation is carried by nucleons moving on two-dimensional toroidal surfaces in such a way as to be anyons.

To find the Kaehler metric on the fermion manifolds we need first to find $\exp(C_{[\Lambda]}\theta)$, which has been treated in some detail by de Wet (1994, 1995, 1996). Specifically

$$e^{\mu\theta} = \mu \sum_{k=0,1,\ldots}^{n} \frac{F_k(\mu)\cos\lambda_k\theta}{i\lambda_k F_k(i\lambda_k)} + i \sum_{k=1,2,\ldots}^{n} \frac{F_k(\mu)\sin\lambda_k\theta}{F_k(i\lambda_k)} \tag{3.10}$$

where μ is an irreducible subspace containing $[\Lambda]$ of $C_{[\Lambda]}$, $\{1; \lambda_2, \ldots; \lambda_n\}$ are normalized positive, distinct, and real eigenvalues of the subspace B of (2.8), and

$$F_0(\mu) = \frac{F(\mu)}{\mu}, \qquad F_k(\mu) = \frac{F(\mu)}{(\mu^2 + \lambda_k^2)}, \qquad F_k(\mu)F_l(\mu) = 0$$

if

$$F(\mu) \equiv \mu(\mu^2 + 1)(\mu^2 + \lambda_2^2)\ldots(\mu^2 + \lambda_m^2) = 0 \tag{3.10a}$$

Writing (3.10) as

$$e^{\mu\theta} = Z_0(\cos\theta) + Z_1(\sin\theta) = \begin{matrix} Z_0 & Z_1 \\ -Z_1 & Z_0 \end{matrix}$$

we can now follow Kobayashi and Nomizu (1969, Chapter IX, §6) and Wong (1967) to find the metric on a complex Grassmann manifold, i.e.,

$$ds^2 = \mathrm{Tr} \frac{dT}{(1 + T\bar{T}^t)} \cdot \frac{d\bar{T}^t}{(1 + T\bar{T}^t)} \tag{3.11}$$

where

$$T = Z_1 Z_0^{-1} = -T^t = \mu \sum_{k=1,2,\ldots}^{n} \frac{i(F_k(\mu)/\mu)\tan\lambda_k\theta}{F_k(i\lambda_k)} \tag{3.11a}$$

$$T\bar{T}^t = \sum_{k=1,2,\ldots}^{n} K_k(\mu)\tan^2\lambda_k\theta \tag{3.11b}$$

Here $\overline{T}^t$ and $d\overline{T}^t$ are the conjugate transposes of T and dT, and

$$K_k(\mu) = i\,\lambda_k F_k(\mu)/F_k\,(i\lambda_k)\mu \tag{3.11c}$$

is idempotent, so that (3.11) reduces to the flat measure carried by a torus, namely

$$ds^2 = \sum_{k=1,2,\ldots}^{n} dz_k\, d\bar{z}_k, \qquad z_k = i\lambda_k\theta \tag{3.12}$$

However, a translation to the normalized canonical form

$$\{0;\ 1;\ \lambda_2, \ldots;\ \lambda_n\}, \qquad n \le p \tag{3.13}$$

where $\{\lambda_2; \ldots; \lambda_n\}$ are all positive, involves adding or subtracting an angular momentum λ_0 and then dividing by $\lambda_t = (\lambda_1 \pm \lambda_0)$, which may be absorbed in θ and does not change the geodesics, although there is a frequency change in the wave function $e^{\mu\sigma}$. Examples of the translation are given in the last columns of Table I.

The effect of the translation is to multiply (3.11a) by $\tan \lambda_0\theta$, which introduces the new distorted metric

$$\begin{aligned} ds^2 &= g_{k\bar{k}}\,(d\lambda_k\theta)\, d(-\lambda_k\theta) \\ &= d(\lambda_k\theta)\, d\,(-\lambda_k\theta) \sum_k \frac{\mu}{\lambda_k} K_k(\mu)\, g(\lambda_k\theta) \sum_k \frac{\overline{\mu}^t}{\lambda_k} K_{\bar{k}}\,(\overline{\mu}^t)\, g\,(-\lambda_k\theta) \end{aligned} \tag{3.14}$$

with

$$g(\lambda_k\theta) = -g(-\lambda_k\theta) = \tan \lambda_0\theta \sec^2\lambda_k\theta/(1 + \tan^2\lambda_0\theta \tan^2\lambda_k\theta)$$

Here $\mu = -\overline{\mu}^t$, $K_{\bar{k}}\,(\overline{\mu}^t) = K_k(\mu)$ and $k = \lambda_k\theta$ are the p distinct coordinates of B in (2.8); $k = -\lambda_k\theta$ and $\pm i\lambda_k$ are the coordinates of μ.

We observe that (3.14) is Einstein according to the definition of Atiyah *et al.* (1978) because only even products of μ occur, which means a diagonal representation like (3.6). In other words, the fermion metric is a solution of the source-free Einstein equations.

This also ensures that the Ricci tensor vanishes, but the sectional curvature

$$K = R_{k\bar{k}k\bar{k}} = \frac{\partial^2 g_{k\bar{k}}}{\partial k \partial \bar{k}} - \sum^{p} \frac{\partial^2 g_{l\bar{l}}}{\partial l \partial l} \tag{3.15}$$

does not, because curvature is determined by the orientation of the remaining p-planes. Thus a spinor field corresponding to the state $[\lambda]_k$ and propagated parallelly only around the section kk will return to its original value, which is precisely the condition found by Green *et al.* (1993, Chapter 15) to show that a Calaba–Yau space or K_3 surface carries a string field.

The sectional curvature K has been calculated by de Wet (1996) for ^{9}Li and its mirror nucleus ^{9}C; a rough sketch is provided by Fig. 2 of that reference. If, however, we regard the nucleus as a canonical ensemble, then the radii of surface curvature should be uniformly distributed, and if we assume that the principal radii of curvature are both equal to R, then to the scales used in Fig. 2,

$$\frac{64}{R} = \sqrt{K} \approx \tan\frac{5\theta}{3}\left(1 - \tan\frac{5\theta}{3}\right)\left[\sum_{k=1,\ldots}^{7} \frac{2\sec^4\lambda_k\theta\,\tan^2\lambda_k\theta}{(1+\tan^2\lambda_k\theta\,\tan^2\frac{5}{3}\theta)^2}\right]^{1/2} \tag{3.16}$$

where

$$\lambda = \{0;\ \tfrac{1}{2};\ \tfrac{5}{6};\ 1;\ \tfrac{4}{3};\ \tfrac{3}{2};\ \tfrac{5}{2};\ 5\}$$

Figure 1 is a plot of $64/R$ versus θ over the region of θ covering the instanton 2 on the surface of ^{9}C, and integration of the density function gives the uniform probability law shown by the straight line (to a different scale). There is a point of inflection very near 375°, which could be the reason for decay to ^{9}B + β^+ after 126 msec when the toroidal manifold becomes an orbifold.

4. CONCLUSION

Expansion of (3.10a) leads to the relation

$$F_k(\mu) = \mu^{2n-1} + \beta_1\mu^{2n-3} + \beta_2\mu^{2n-5} + \ldots + \beta_n\mu \tag{4.1}$$

where $\mu^3 \to -\mu^3$, $\mu^5 \to \mu^5$, $\mu^7 \to -\mu^7$, etc., by (2.8) and

$$\beta_1 = \lambda_1^2 + \lambda_2^2 + \ldots + \hat{\lambda}_k^2 + \ldots. + \lambda_n^2 \qquad (\lambda_k = 0) \tag{4.1a}$$

$$\beta_2 = \lambda_1^2\,\lambda_2^2 + \lambda_1^2\lambda_3^2 + \ldots + \lambda_1^2\lambda_n^2 + \ldots\,\lambda_{n-1}^2\,\lambda_n^2 \qquad (\lambda_k = 0)\,\vdots \tag{4.1b}$$

$$\beta_n = \lambda_1^2\,\lambda_2^2 \ldots \hat{\lambda}_k^2 \ldots \lambda_n^2 = \mathrm{Det}(\mu_k) \qquad (\lambda_k = 1) \tag{4.1n}$$

Equation (4.1n) may be recognised as the Slater determinant of the antisymmetrized combination of all one-fermion orbits excluding the state $[\lambda]_k$. If the particles are independent, there are $A!$ ways of distributing A fermions among each configuration of the A orbits $\nu_1\nu_2 \ldots \nu_A$; however, in our case $[\lambda]_k$ is a coherent state consisting of $N_{[\lambda]_k} = A!/(\lambda_1!\ \lambda_2!\ \lambda_3!\ \lambda_4!)$ configuration coordinates with the same net values of spin, parity, and isospin. In fact the $[\lambda]_k$ are fibers whose delineation yields the rotational character of μ, so $F_k(\mu)$ is a sum of correlations of many-particle and rotational states such that the number of possible many-particle states is restricted by rotational possibilities. In particular when the Slater determinant is unity, signifying that all the orbits $\nu_1\nu_2 \ldots \nu_k$ are filled, μ^{2n-1} is a measure of all the rotational possibilities of A spinning nucleons moving around one another and is heavily dependent upon the number n of rotational eigenvalues.

In between these extremes, β_i is a principal minor of μ and we may consider β_n as a primary correlation, β_{n-1} as a secondary correlation, β_{n-2} a ternary correlation, and so on. This corresponds to the construction of subdynamics by Prigogine (1995) and his co-workers (e.g., Antoniou and Tasaki, 1993) which also depends on a density function approach.

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